Graph

Introduction

Graphs are used in a wide variety of models with computer science such as communication network, logical design, transportation networks, formal languages, compiler writing and retrieval. For example: in a communication network, where computers can be represented by vertices and communication links by edges.

Outlines

- ❑ Definition of Graph.
- ❑ Basic types of graphs
- ❑ Basic Terminology.
- ❑ Representation of Graph.
    - Adjacency List.
    - Adjacency Matrix.

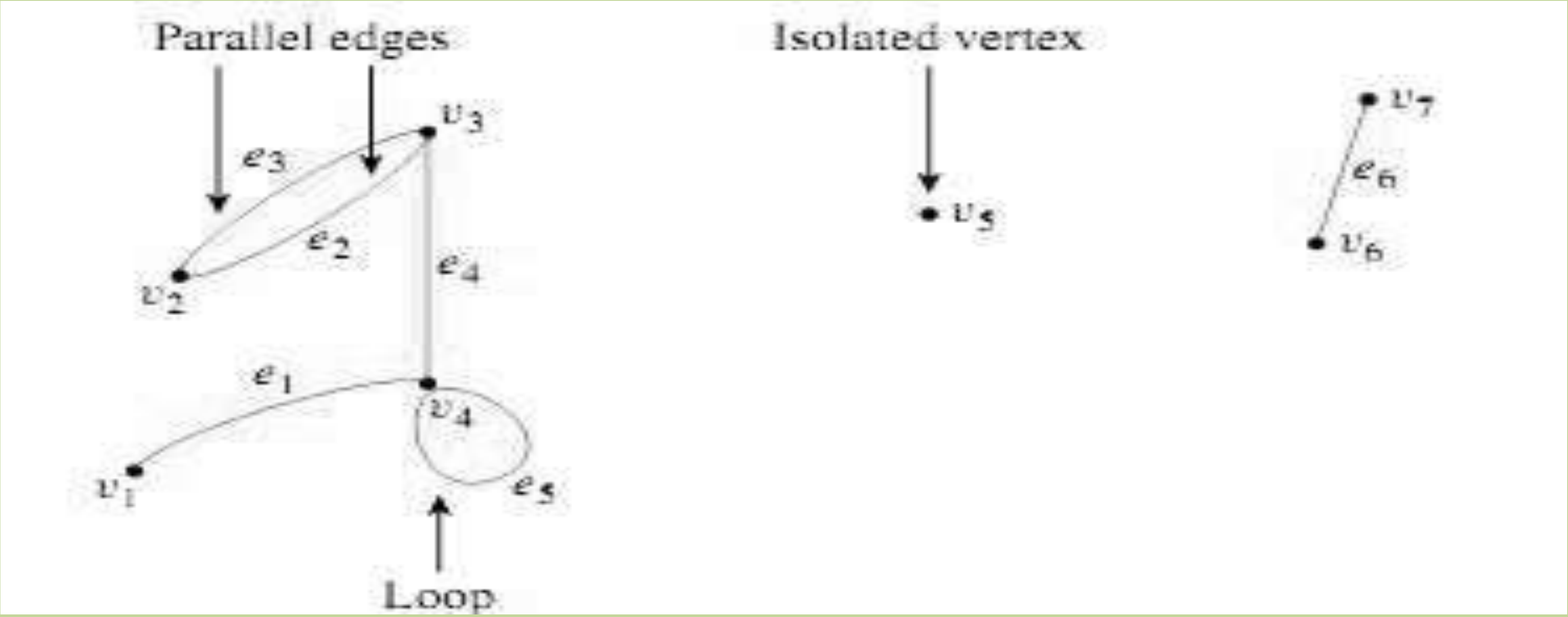
Definition of Graph

Graphs are discrete structures consisting of vertices and edges that connect these vertices, so a graph G(V,E) consists of:

V , a nonempty set of vertices (or nodes) and
E, a set of edges. Each edge has either one or two vertices associated with it, called its endpoints.

In general, we visualize graphs by using points to represent vertices and line segments, possibly curved, to represent edges.

The edges should either connect one vertex to another or a vertex to itself. See the following figure.



From the figure, can notice the following:

- The vertices have been labeled with (v) and the edges with (e).
- An edge connects a vertex to itself (as e5 does), it is called a loop.
- Two edges connect the same pair of vertices (as e2 and e3 do), they are said to be parallel (or multiple).
- The vertex that is unconnected by an edge to any other vertex in the graph (as v5 is), is called isolated.

The graph can be defined formally by specifying its vertex set, its edge set, and the edge-endpoint table. The description of graph shown in the figure is as following:

- vertex set = {v1, v2, v3, v4, v5, v6, v7}.
- edge set = {e1, e2, e3, e4, e5, e6}.
- edge-endpoint table:

edge	endpoints
e_1	$\{v_1, v_4\}$
e_2	$\{v_2, v_3\}$
e_3	$\{v_2, v_3\}$
e_4	$\{v_3, v_4\}$
e_5	$\{v_4, v_4\}$
e_6	$\{v_6, v_7\}$

Basic types of graphs:

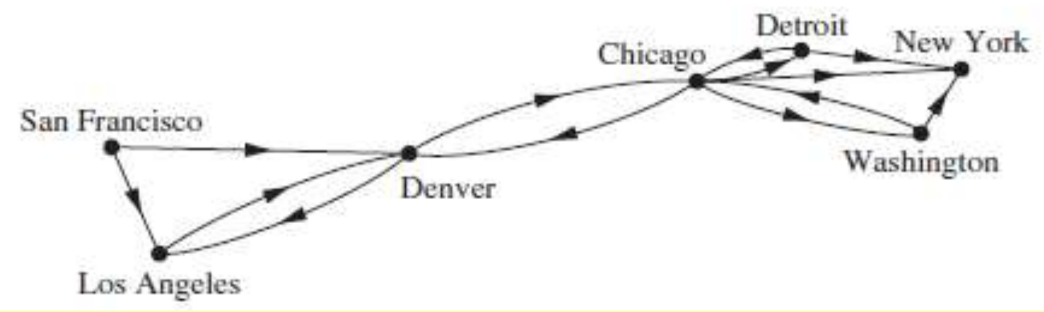
1. Directed Graph (Digraph)

A directed graph (or digraph) G= (V, E) consists of a nonempty set of vertices V and a set of directed edges (or arcs) E. Each directed edge is represented by an ordered pair of vertices. The directed edge represented by (u, v) is said to start at u and end at v.

An arrow pointing from u to v used to indicate the direction of an edge that starts at u and ends at v. The vertex u is called the initial vertex of (u, v), and v is called the terminal or end vertex of (u, v).

A directed graph may contain

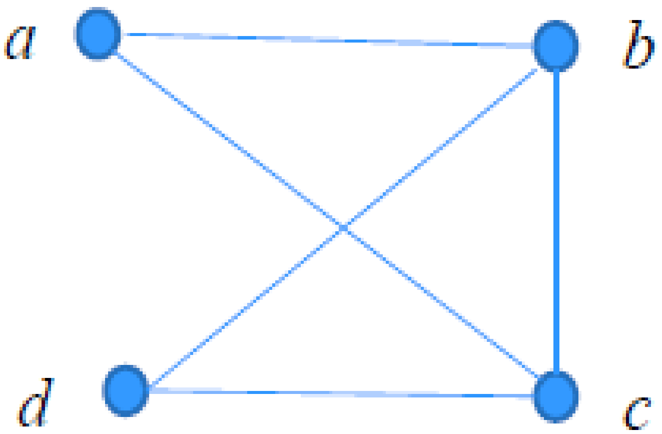
- Loops.
- Multiple directed edges that start and end at the same vertices.
- Directed edges that connect vertices u and v in both directions.



2. Undirected graphs

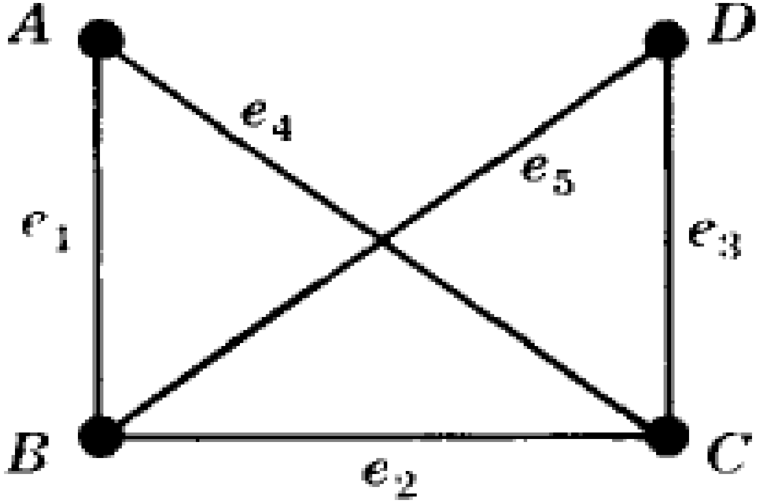
An undirected graph is graph, i.e., a set of objects (called vertices or nodes) that are connected together, where all the edges are bidirectional. An undirected graph is sometimes called an undirected network.

undirected graph is defined in term of unordered pairs of vertices.



Basic Terminology

Adjacent Vertices (Neighbors)



V consists of four vertices A, B, C, D ; and, E consists of five edges

e1 ={A,B},

e2 = {B,C},

e3 = {C, D},

e4 = {A, C},

e5 = {B, D}.

Two vertices u and v in an undirected graph G are called adjacent (or neighbors) if u and v are endpoints of an edge.

When (u, v) is an edge of the graph G with directed edges, u is said to be adjacent to v, but v is not adjacent to u.

Degree:

In an undirected graph, The degree of a vertex (denoted by deg (v)) is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The sum of the degrees of the vertices of a graph is equal to twice the number of edges.

$2m = \sum_{v \epsilon V} \deg(v)$, with m edges. (Note that this applies even if multiple edges and loops are present.)

In a digraph, the degree of a vertex is the sum of its in-degree and out-degree.

The in-degree of a vertex (denoted by deg^{-}(v)) is the number of edges coming to the vertex.

The out-degree of a vertex (denoted by deg^{+} (v)) is the number of edges leaving the vertex.

(Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex).

Let *G* = *(V ,E)* be a graph with directed edges. Then

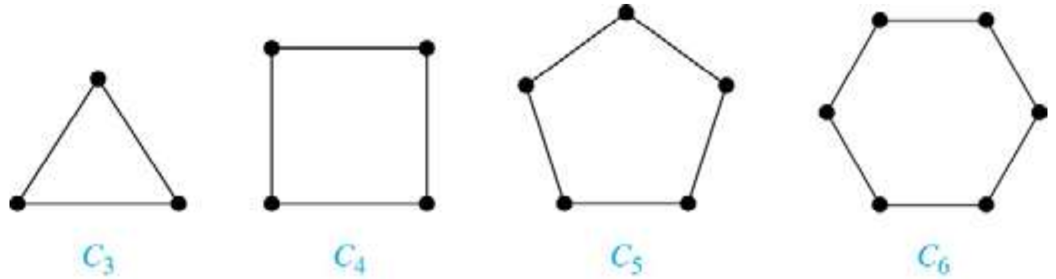
$$|E| = \sum deg^{-}(v) = \sum deg^{+}(v)$$

Path:

The path is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph. The length of a path is the number of edges on the path.

cycle

The path is a cycle if it begins and ends at the same vertex, that is, if u = v, and has length greater than zero.

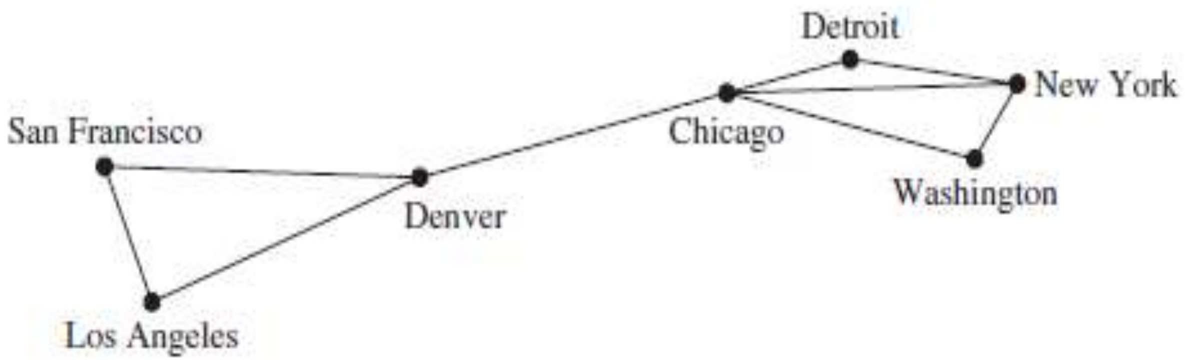


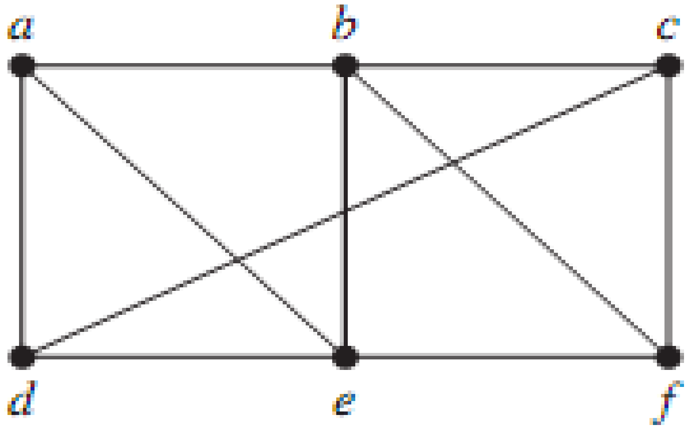
Simple Path:

A path or circuit is simple if it does not contain the same edge more than once.

Simple Graph:

A graph in which each edge connects two different vertices. it does not have any loops or parallel edges.

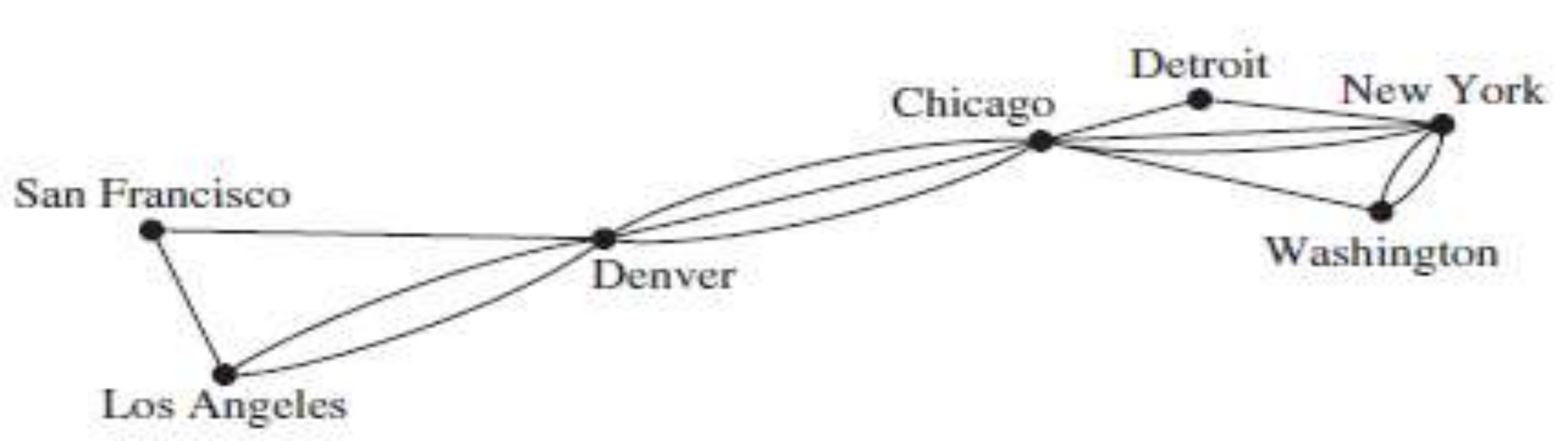




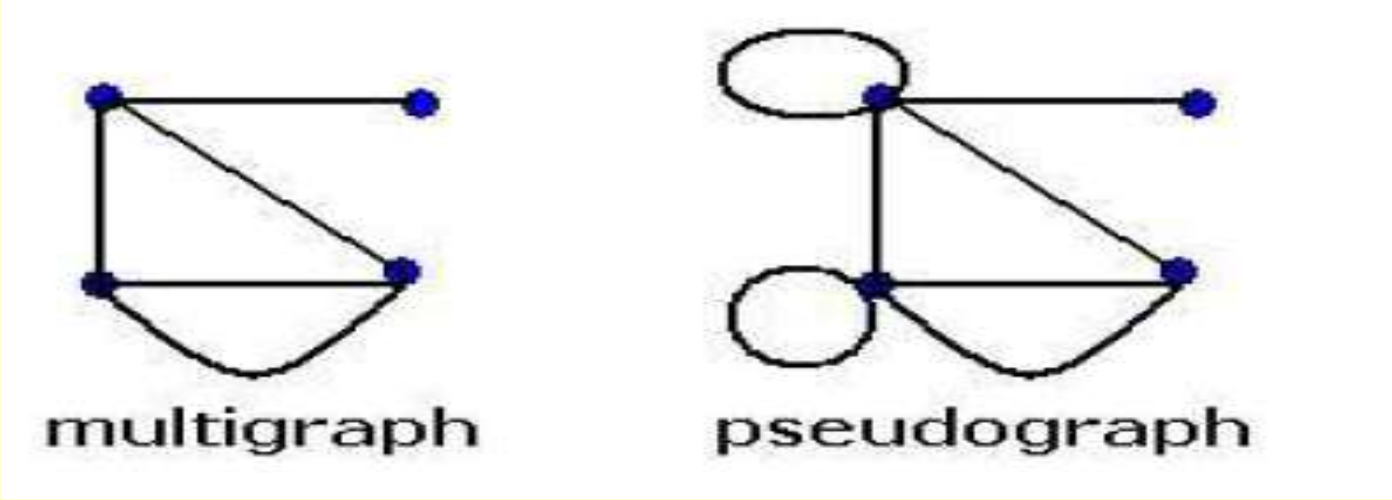
- From the simple graph shown in the Figure, notice the following:

- a, d, c, f, e: is a simple path of length 4, because {a,d}, {d,c}, {c, f}, and {f, e} are all edges.
- d, e, c, a: is not a path, because {e, c} is not an edge.
- b, c, f, e, b: is a circuit of length 4 because {b, c}, {c, f}, {f, e}, and {e, b} are edges and this path begins and ends at b.
- a, b, e, d, a, b:is a path of length 5, is not simple because it contains the edge {a,b} twice.

- Multigraphs:
- A graph that have loops, multiple edges connecting the same vertices.
- allow multiple edges between vertices.

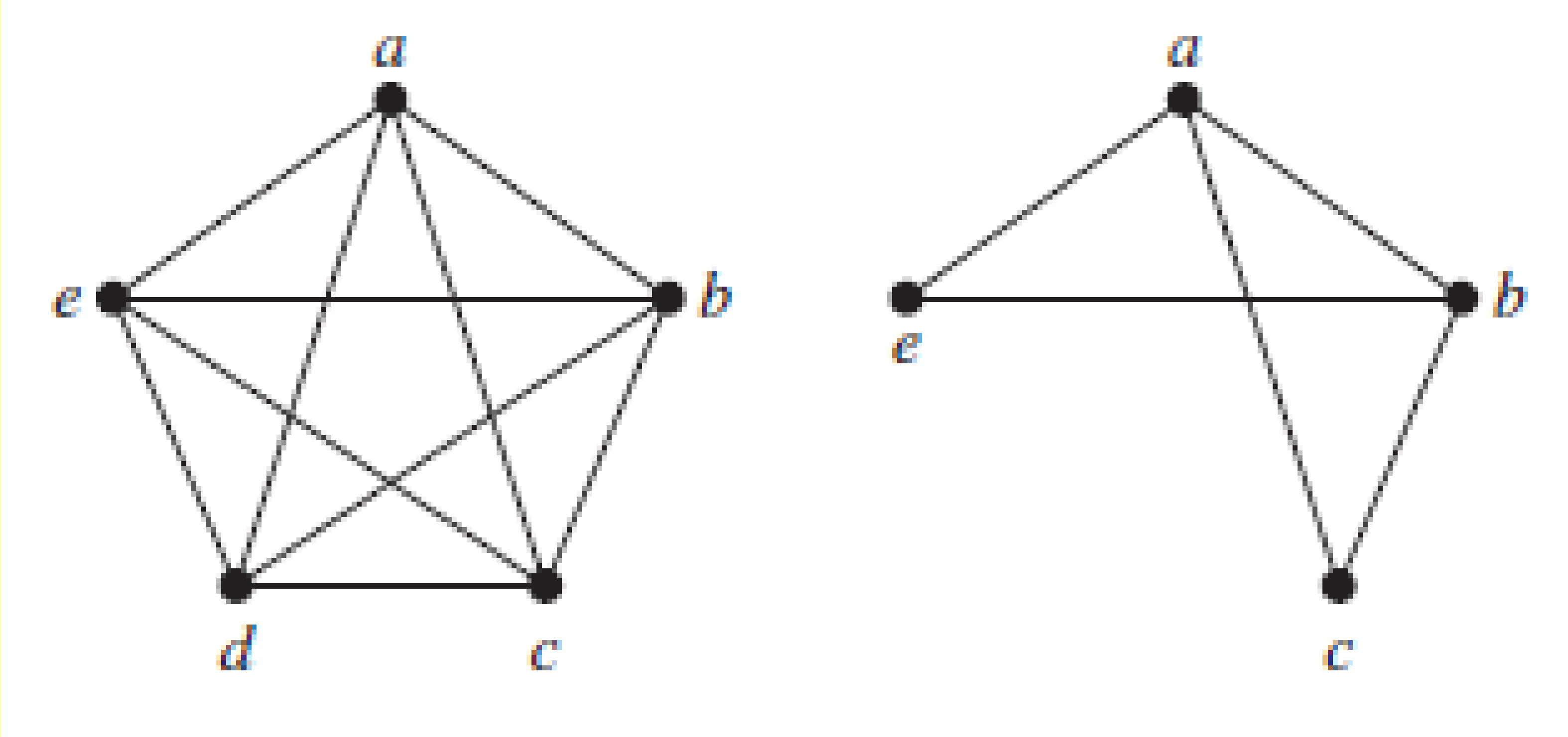


Pseudographs: allow edges connect a vertex to itself.

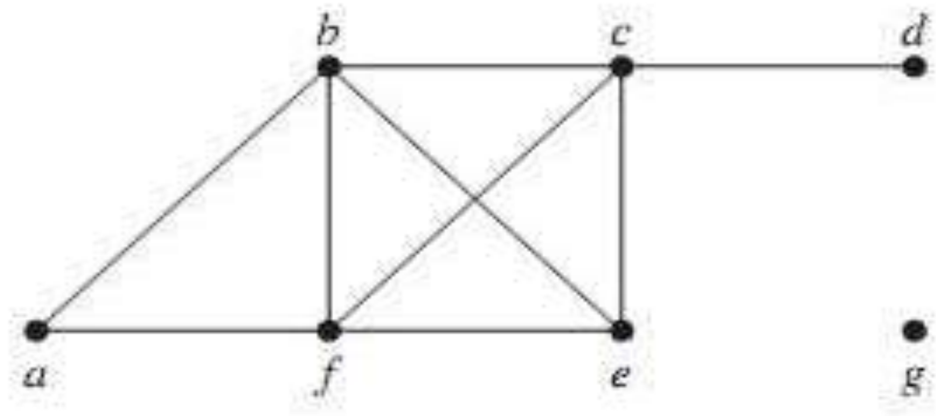


Subgraphs

Consider a graph G = G(V,E) and a graph H = H(V', E') is called a subgraph of G if the vertices and edges of H are contained in the vertices and edges of G, that is, if $V' \subseteq V$ and $E' \subseteq E$.



Ex1: What are the degrees and neighbors of the vertices in the undirected graph G.



G

Sol: Graph G

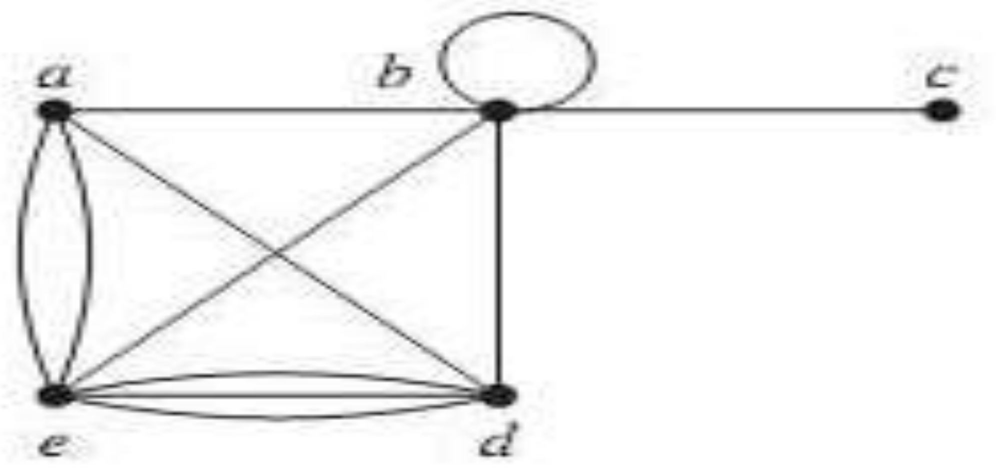
Degree: deg(a) = 2, deg(b) = 4, deg(c) = 4, deg(d) = 1, deg(e) = 3, deg(f) = 4, deg(g) =0;

Neighbors: N(a) = {b, f} , N(b) = {a, f, e, c}, N(c) = {b, f, e}, N(d) = {c}, N(e) = {c, b, f},
N(f) = {e, c, b, a}, N(g)= {}

Graph G is simple graph.

Sum of degree= 18

Ex1: What are the degrees and neighbors of the vertices in the undirected graph H displayed below.



H

Graph H :Degree: deg(a) = 4, deg(b) = 6, deg(c) = 1, deg(d) = 5, deg(e) = 6.
Neighbors: N(a) = {e, d, b} , N(b) = {a, b, c, d, e}, N(c) = {b}, N(d) = {b, a, e}, N(e) = {a, b, d}.

Graph H is multigraph.

Ex: How many edges are there in a graph with 10 vertices each of degree six?

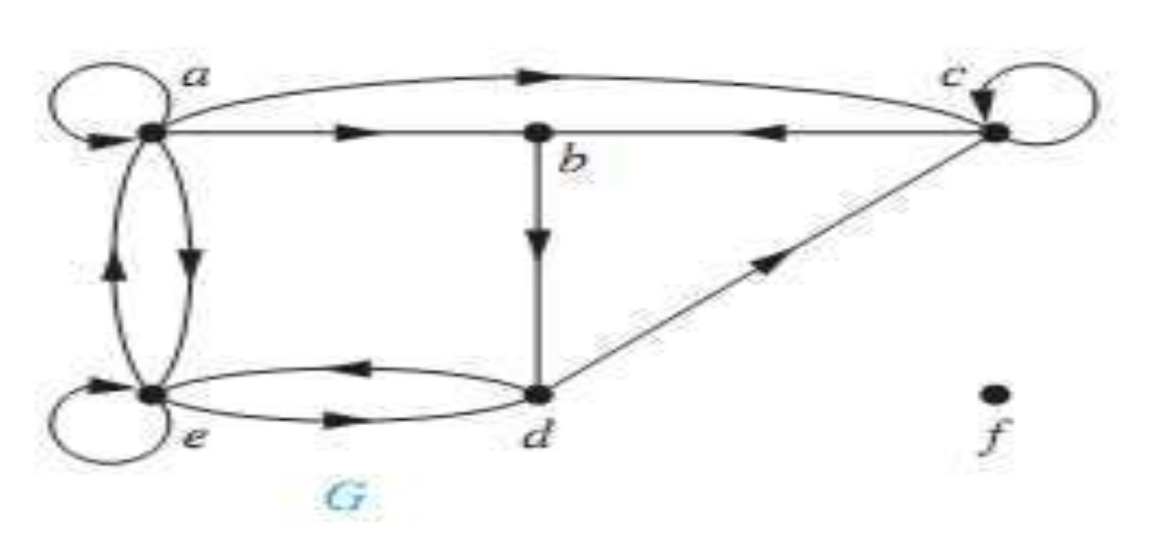
Solution:

Because the sum of the degrees of the vertices is $6 * 10 = 60$, it follows that $2m = 60$

where m is the number of edges.

Therefore, $m = 30$.

Ex: Find the in-degree and out-degree of each vertex in the digraph G shown below.



Sol:

in-degrees in G: deg^{-} (a) = 2, deg^{-}(b) = 2, deg^{-}(c) = 3, deg^{-}(d) = 2, deg^{-}(e) = 3, deg^{-}(f) =0.

out-degrees in G: deg^{+} (a) = 4, deg^{+} (b) = 1, deg^{+} (c) = 2, deg^{+} (d) = 2, deg^{+} (e) = 3, deg^{+} (f) =0.

Representation of Graph

The two common ways to represent graphs are:

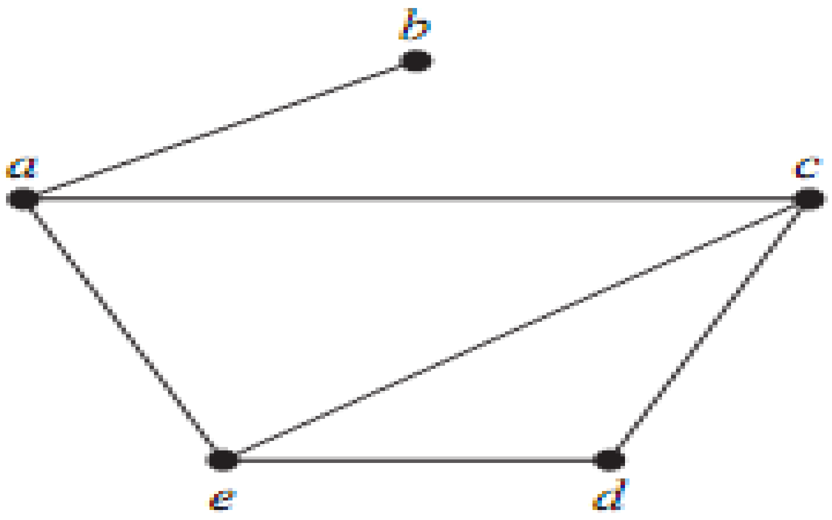
1- Adjacency lists.

2- Adjacency matrix.

1- Adjacency List

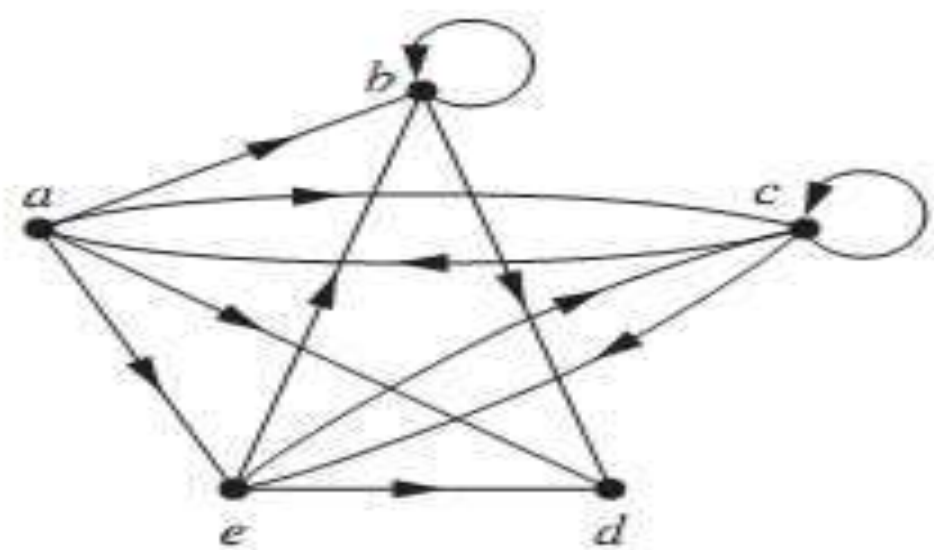
The adjacency list represent the graph by specify the vertices that are adjacent to each vertex of the graph. Directed graph represented by listing all the vertices that are the terminal vertices of all vertices of the graph.

Ex : Use adjacency lists to describe the simple graph given in Figure below:



Vertex	Adjacent Vertices
a	b, c, e
b	A
c	a, e, d
d	c, e
e	a, c, d

Ex : Use adjacency lists to describe the digraph given in Figure below:



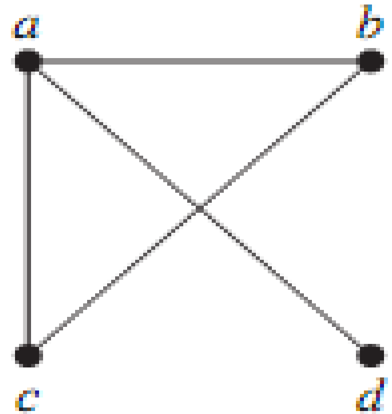
Initial Vertex	Terminal Vertices
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

2- Adjacency Matrix

The adjacency matrix A of a graph is the n x n zero–one matrix with 1 as its (i, j)th entry when vi and vj are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words, if its adjacency matrix is A =[aij], then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \text{otherwise.} \end{cases}$$

Ex : Use an adjacency matrix to represent the graph shown below.



Sol:

S is not reflexive: There is no loop at 0

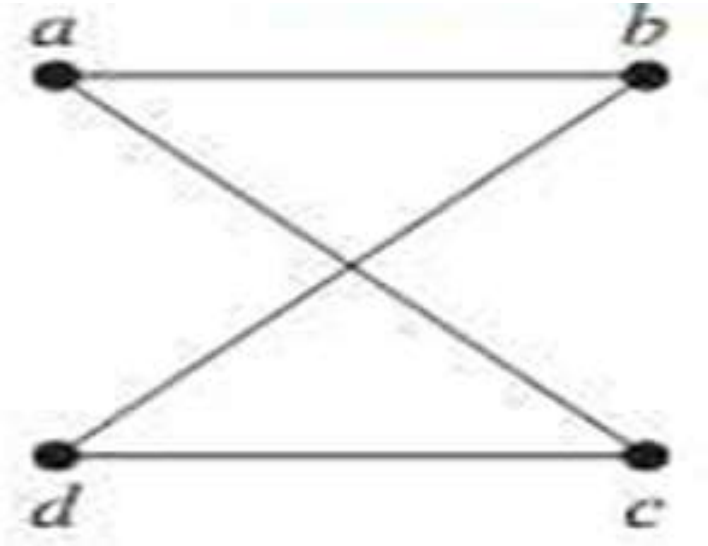
S is not symmetric: There is an arrow from 0 to 1 but not from 1 to 0.

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Ex: Draw a graph with the following adjacency matrix, with respect to the ordering of vertices a, b, c, d.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Sol:



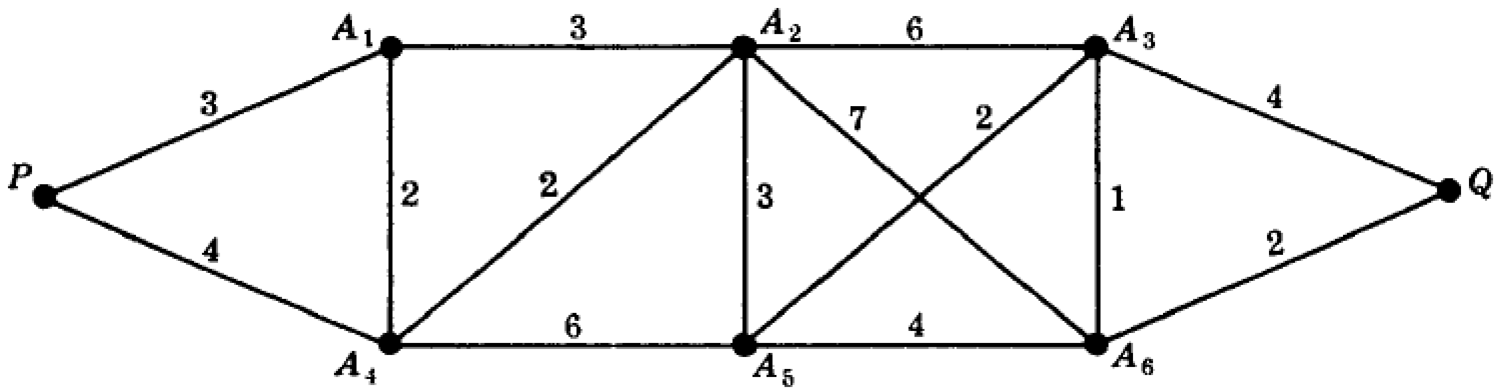
Note: In the undirected graph, when multiple edges connecting the same pair of vertices vi and vj ,or multiple loops at the same vertex, are present, the adjacency matrix is no longer a zero–one matrix, because the (i, j)th entry of this matrix equals the number of edges that are associated to {vi , vj}.

Labeled and Weighted graphs

A graph G is called a labeled graph if its edges and/or vertices are assigned data. If each edge (e) is assigned a non-negative number L(e). Then L(e) is called the weight or length of e. The weight of a path in such a weighted graph G is defined to be the sum of the weights of the edges in the path.

One important problem in graph theory is to find a shortest path, that is, a path of minimum weight (length), between any two given vertices.

Example: find the minimum path between P & Q:



(P, A1, A2, A5, A3, A6, Q)

Q

$$\sum_{p}^{Q} \text{L (e)} = 3 + 3 + 3 + 2 + 1 + 2 = 14$$

Another minimum path:

(P, A4, A2, A5, A3, A6, Q)

$$\sum_{p}^{Q} \text{L (e)} = 4 + 2 + 3 + 2 + 1 + 2 = 14$$